Estimates of the transition density of a gas system

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Abstract.
Let X be the diffusion Markov process on $\mathbf{R}^d$ with the generator $L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^d b_i(x)\partial_{x_i}$, and transition density $G(t,x,y)$. Under some conditions on the matrix $a(x)$ we get the estimate
$\sup_{0<t<T, x,y\in\mathbf{R}^d} \sqrt{t}\frac{|\nabla_x G(t,x,y)|}{G(2t,x,y)} < +\infty$ for all $T>0$.
The latter estimate is used to get the existence and uniqueness of a solution of the following gas system

$$\mathcal{S}(a,b)\begin{cases} \partial_t(\rho) + div(u\rho) = L^*(\rho) \\ \partial_t(u_i\rho) + div(u_i u\rho) = L^*(u_i\rho), \\ \forall\, 1\le i\le d,\ \rho(dx,t)\to\rho_0(dx), u_i(x,t)\rho(dx,t)\to v_i(x)\rho_0(dx) \\ \text{weakly, as } t\to 0^+ \end{cases}$$

where $\rho_0(dx)$ (a probability measure on $\mathbf{R}^d$), and the bounded vector field $v := (v_1,...,v_d) : \mathbf{R}^d \to \mathbf{R}^d$ are given. The family of probability measures $\rho := \rho(dx,t)$ and the velocities $u := u(x,t)$ are unknown. Here L^* is the formal adjoint operator of L.

Résumé
Soit X un processus de Markov à valeurs dans $\mathbf{R}^d$, de générateur $L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^d b_i(x)\partial_{x_i}$, et de densités de probabilités de transition $G(t,x,y)$.
On obtient, sous certaines conditions sur la matrice $a(x)$, l'estimation
$\sup_{0<t<T, x,y\in\mathbf{R}^d} \sqrt{t}\frac{|\nabla_x G(t,x,y)|}{G(2t,x,y)} < +\infty$ pour tout $T>0$. On utilise cette dernière estimation pour obtenir l'existence et l'unicité du système de gaz $\mathcal{S}(a,b)$ ci-dessus.

Keywords. Transition density, Aronson estimates, Gas system, Propagation of chaos.

1 Introduction and the main results

We suppose that a_{ij}, b_i belong to the set $C_b^{2+\alpha}(\mathbf{R}^d)$ of bounded functions f which have two bounded derivatives, and $\partial^2_{x_i x_j} f$ are Hölder continuous with exponent $\alpha \in (0,1)$. The matrix $a(x)$ is symmetric and $a(x) \geq \lambda I_{d\times d}$, where $\lambda > 0$ and $I_{d\times d}$ is the $d \times d$ identity matrix. The transition density $G(t,x,y)$ is the fundamental solution of the parabolic equation

$$\partial_t G(t,x,y) = LG(t,x,y), t > 0, x \in \mathbf{R}^d$$

and $G(0,x,y) = \delta(x-y)$.

Let $g(x)$ be the inverse of the matrix $a(x)$. We set $\lambda_g = \inf_{|\xi|=1}\langle g(x)\xi, \xi\rangle$ and $\Lambda_g = \sup_{|\xi|=1}\langle g(x)\xi, \xi\rangle$.

Now we can announce our first result.

Theorem 1.1. 1) If $2\lambda_g > \Lambda_g$, then for small $T > 0$ there exists $c > 0$ such that for all $1 \leq i \leq d$,

$$|\partial_{x_i} G(t,x,y)| \leq \frac{c}{\sqrt{t}} G(2t,x,y) \quad \forall\, 0 < t \leq T, x, y \in \mathbf{R}^d \tag{1}$$

2) If $2\lambda_g^2 > \Lambda_g^2$, then (1) works for all T.

Our second result is the following:

Theorem 1.2. We suppose that $\rho_0(dx) = \rho_0(x)dx$, with $\rho_0 \in L^2(\mathbf{R}^d, dx)$. Under the latter hypothesis the following assertions hold.

1) The system $\mathcal{S}(a,b)$ has a weak solution in the set $\mathcal{U}_{||v||_\infty} \times C(\mathbf{R}_+, M(\mathbf{R}^d))$, of measurable velocity u bounded by $||v||_\infty$ and $(t \in \mathbf{R}_+ \to \rho(dx,t)) \in C(\mathbf{R}_+, M(\mathbf{R}^d))$. Here $M(\mathbf{R}^d)$ is the set of probability measures on $\mathbf{R}^d$.

2) Under the first hypothesis of Theorem 1.1, the system $\mathcal{S}(a,b)$ has a unique weak solution in the set $\mathcal{U}_{||v||_\infty} \times C([0,T], M(\mathbf{R}^d))$ for a small interval of time $[0,T]$.

3) Under the second hypothesis of Theorem 1.1, the system $\mathcal{S}(a,b)$ has a unique weak solution in the set $\mathcal{U}_{||v||_\infty} \times C(\mathbf{R}_+, M(\mathbf{R}^d))$.

We refer to [4], [5], [7], [6] and references herein, for links between $\mathcal{S}(a,b)$ and pressureless gas equations.

The plan of the rest of this work is the following. In Section 2 we prove Theorem 1.1. In Section 3 we construct a weak solution of $\mathcal{S}(a,b)$, and Section 4 we prove the uniqueness.

2 Proof of Theorem 1.1

Before giving the proof we need auxiliary bounds for the transition density $G(t,x,y)$. In the theory of partial differential equations and stochastic differential equations the following estimates

$$K_2 t^{-d/2}\exp(-c_2\frac{|x-y|^2}{t}) \le G(t,x,y) \le K_1 t^{-d/2}\exp(-c_1\frac{|x-y|^2}{t})$$

are well known. [1], [2], [8], [9], [19]. On the other hand, the following result was obtained in [17] by using the idea of Fleming's logarithmic transformation [3]:

$$\frac{1}{\sqrt{deta(y)}(2t\pi)^{d/2}}k_2(t)\exp(-c_2(t)I_b(t,x,y)) \le G(t,x,y) \tag{2}$$

and

$$G(t,x,y) \le \frac{1}{\sqrt{deta(y)}(2t\pi)^{d/2}}k_1(t)\exp(-c_1(t)I_b(t,x,y)) \tag{3}$$

and for all $1 \le i \le d$,

$$|\partial_{x_i}\ln(G(t,x,y))| \le \frac{c}{\sqrt{t}}(I_b(t,x,y)+1)^{1/2} \tag{4}$$

where $k_1, k_2 > 0$ are bounded above and bounded below away from 0 on bounded intervals $(0,T]$. Here

$$I_b(t,x,y) = \inf\{\frac{1}{2}\int_0^t \sum_{i,j} g_{ij}(\varphi(s))(\dot\varphi(s)-b(\varphi(s))_i(\dot\varphi(s)-b(\varphi(s))_j ds\}$$

the infimum is taken under $\varphi \in H^1([0,t],\mathbf{R}^d)$ such that $\varphi(0)=x, \varphi(t)=y$. See also [20], [13] for the estimate (4).

Proof of Part 1. First we remark that (1) is equivalent to say that for all $T>0$ there exists $c>0$ such that for all $1 \le i \le d$, $0 < t \le T, x, y \in \mathbf{R}^d$

$$|\partial_{x_i}(\ln(G(t,x,y))|\frac{G(t,x,y)}{G(2t,x,y)} \le \frac{c}{\sqrt{t}}$$

Thanks to (4) a sufficient condition to get the latter estimate is

$$I_b(t,x,y)^{1/2}\frac{G(t,x,y)}{G(2t,x,y)} \le c \quad \forall\, 0 < t \le T, x, y \in \mathbf{R}^d \tag{5}$$

We are going to estimate $I_b(t,x,y)$ by $I_0(t,x,y)$. We have

$$g_{ij}(\varphi(t))(\dot\varphi(t)-b(\varphi(t))_i(\dot\varphi(t)-b(\varphi(t))_j =$$

$$g_{ij}(\varphi(t))\dot{\varphi}_i(t)\dot{\varphi}_j(t) - g_{ij}(\varphi(t))\dot{\varphi}_i(t)b_j(\varphi(t))$$
$$+g_{ij}(\varphi(t))b_i(\varphi(t))b_j(\varphi(t))$$

From that and from Cauchy Schwarz inequality we have

$$I_0(t,x,y) - ||b||_\infty \sqrt{t} I_0^{1/2}(t,x,y) - \Lambda_g t ||b||_\infty^2 \leq$$

$$I_b(t,x,y) \leq I_0(t,x,y) + ||b||_\infty \sqrt{t} I_0^{1/2}(t,x,y) + \Lambda_g t ||b||_\infty^2$$

We see easily that for all $\delta > 0$ there exists $c > 0$ such that $\alpha^{1/2} \leq \delta\alpha + c$ for all $\alpha > 0$. From that we have for all $\varepsilon > 0$ the existence of $c > 0$ such that

$$(1-\varepsilon)I_0(t,x,y) - ct \leq I_b(t,x,y) \leq (1+\varepsilon)I_0(t,x,y) + ct$$

From that and from (2), (3) we have

$$\frac{1}{\sqrt{deta(y)}(2\pi t)^{d/2}} k_2(t) \exp(-c_2(t)(1+\varepsilon)I_0(t,x,y)) \leq G(t,x,y) \tag{6}$$

and

$$G(t,x,y) \leq \frac{1}{\sqrt{deta(y)}(2\pi t)^{d/2}} k_1(t) \exp(-c_1(t)[(1-\varepsilon)I_0(t,x,y)]) \tag{7}$$

On the other hand we have

$$I_0(2t,x,y) = \frac{1}{2} I_0(t,x,y) \tag{8}$$

Now from (8), (6), (7) a sufficient condition to have (5) is

$$I_b(t,x,y)^{1/2} \exp([c_2(2t)(1+\varepsilon)I_0(2t,x,y) - c_1(t)(1-\varepsilon)I_0(t,x,y)]) < c$$

for all $t < T, x, y \in \mathbf{R}^d$. It is equivalent to say that there exists $k_T > 0$, such that

$$[c_2(2t) - 2c_1(t)] < -k_T < 0 \tag{9}$$

for all $0 < t \leq T, x, y \in \mathbf{R}^d$.

It appears from the proof of Theorem A in [17] that

$$c_1(t) = \delta(t) + \lambda_g \Lambda_g^{-1} \tag{10}$$

and

$$c_2(t) = 1 + c(||a||_\infty, |g_{x_i x_j}|_\infty)\lambda_g^{-1} t \tag{11}$$

where the term $\delta(t)$ can be chosen uniformly for $t \in (0,T]$ as small as we want. It follows that (9) is satisfied if and only if

$$2\lambda_g \Lambda_g^{-1} > 1$$

and for small T, which achieves the proof of Part 1 of Theorem 1.1.

For the proof of Part 2 we use the same proof of the lower bound of $-\ln(G(t,x,y))$ in ([17], Theorem A pages 547-550) and we get an upper bound of $-\ln(G(t,x,y)$. The latter upper bound gives a lower bound for $G(t,x,y)$ similar to (6) with

$$c_2(t) = \Lambda_g \lambda_g^{-1} + \delta(t), \tag{12}$$

where $\delta(t)$ can chosen uniformly for $t \in (0,T]$ as small as we want. Using the new form of $c_2(t)$ in (9) we get the condition

$$\lambda_g \Lambda_g^{-1} - \frac{\lambda_g^{-1}\Lambda_g}{2} > 0$$

which achieves the proof of Theorem 1.1.

3 Existence of a weak solution

The construction is nearly the same as in [4]. The main idea is the construction of the following nonlinear stochastic differential equation:

$$dX_t = (\mathbf{E}[\, v(X_0) \,|\, X_t\,] + b(X_t))dt + \sigma(X_t)dB_t, \quad \mathcal{L}(X_0) = \rho_0(dx) \tag{13}$$

where B_t is a standard Brownian motion independent of X_0, and $\sigma(x)$ is the unique nonnegative square root of $a(x)$. Having a solution (X_t) of (13) we show using Itô's formula that $(\rho(dx,t) := P(X_t \in dx), u(x,t) = \mathbf{E}[v(X_0) \,|\, X_t = x] : t \geq 0)$ is a weak solution of our system $\mathcal{S}(a,b)$. In fact from Itô's formula the process

$$f(X_t) - f(X_0) - \int_0^t (u(X_s,s)\nabla(f)(X_s) + L(f)(X_s))ds \tag{14}$$

is a martingale for all smooth function f. By operating the expectation on the latter equality we get the first equation of $\mathcal{S}(a,b)$. By multiplying the equality (14) by $v_i(X_0)$, and operating the expectation on each member we get the equations of $\mathcal{S}(a,b)$ for $1 \leq i \leq d$.

The construction of (13) is based on the conditional propagation of chaos [21]. The sketch of the proof is the following.

Step 1. We consider the system

$$dX_t^{i,N,n} = \sigma(X_t^{i,N,n}, t)dB_t^i + (\frac{\sum_{j\neq i} v(X_0^j)\varphi^n(X_t^{i,N,n} - X_t^{j,N,n})}{\sum_{j\neq i}\varphi^n(X_t^{i,N,n} - X_t^{j,N,n})} + b(X_t^{i,N,n}))dt \tag{15}$$

here $i = 1, ..., N$, and $(B^i : 1 \leq i \leq N)$ are N independent d-dimensional Brownian motions,

$$\varphi^n(x) = n^d\varphi(nx)$$

and φ is a smooth symmetric probability density on $\mathbf{R}^d$. The initial positions $(X_0^1, ..., X_0^N)$ are *i.i.d.* with probability distribution ρ_0, and they are independent of $(B^1, ..., B^N)$.

In the first stage we keep n fixed and we let $N \to +\infty$. We get by the conditional propagation of chaos already used in [4, 21] a weak solution of the following non-linear diffusion:

$$dX_t^n = \sigma(X_t^n, t)dB_t + (\frac{\int\int v(y)\varphi^n(X_t^n - z)\rho_{0,t}^n(y,z)dydz}{\int\varphi^n(X_t^n - z)\rho_t^n(z)dz} + b(X_t))dt \tag{16}$$

where $\rho_{0,t}^n$ is the probability distribution of (X_0, X_t^n), and ρ_t^n is the probability distribution of X_t^n.

Step 2. In the second stage we let $n \to +\infty$ in (16) and we get a weak solution for (13). The tools of the latter stage are as in [4] and are the following.

Lemma 3.1. ([18], Lemma 11.4.1)

Let (f_n) be a sequence of non-negative $\mathcal{B}(\mathbf{R}^r)$-measurable functions such that $\int f_n(x)dx = 1$ and

$$\lim_{h\to 0}\sup_{n\geq 1}\int |f_n(x+h) - f_n(x)|dx = 0.$$

Assume that there is an $f \in L^1(\mathbf{R}^r)$ such that

$$\int f(x)\psi(x)dx = \lim_{n\to+\infty}\int f_n(x)\psi(x)dx$$

for all $\psi \in C_b(\mathbf{R}^r)$. Then $f_n \to f$ in $L^1(\mathbf{R}^r)$.

Lemma 3.2. ([18], Lemma 9.1.15)

Let c be a bounded measurable functions from $\mathbf{R}^d \times \mathbf{R}_+ \to \mathbf{R}^d$, and $dX_t = c(X_t, t)dt + \sigma(X_t)dB_t$ a diffusion. Then there is a non-decreasing function $\psi : (0, +\infty) \to (0, +\infty)$ depending on d, T, K such that $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$\int_s^T \int |p(s,x,t,y+h) - p(s,x,t,y)| dydt \leq \psi(|h|),$$

where $p(s,x,t,y)$ are the transition probability density determined by the diffusion X.

But Lemma 9 in [4] must be replaced by:

Lemma 3.3. ([15], Chapitre 3)

Let us assume $a_0^i(x,t), a_{ij}(x,t), \nabla_x a_{ij}(x,t) \in L^\infty([0,T] \times \mathbf{R}^d)$. The parabolic equation

$$\frac{du}{dt} = \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x j}(a_{ij}u) - div(a_0 u),$$

with initial value $u_0 \in L^2(\mathbf{R}^d, dx)$ has a unique solution in $L^2([0,T], H^1)$. Here $H^1 = \{f, \nabla f \in L^2(\mathbf{R}^d, dx)\}$. Moreover $\frac{du}{dt} \in L^2([0,T], H^{-1})$.

That is why we weed the condition $\rho_0(dx) = \rho_0(x)dx$ with $\rho_0 \in L^2(\mathbf{R}^d, dx)$ in Theorem 1.2.

4 Uniqueness

4.1 Auxiliary results

We give first some estimates which will be carried through the sequel to get the uniqueness.

Proposition 4.1. Suppose that for all $0 < t < T$ there exists $c > 0$ such that

$$|\nabla_x G(t,x,y)| \leq ct^{-1/2} G(2t,x,y) \tag{17}$$

for all $x, y \in \mathbf{R}^d$.

Let B be a bounded measurable function from $\mathbf{R}^d$ to $\mathbf{R}^d$, and $t \to \rho_t$ a family of probability measures on $\mathbf{R}^d$ which is a weak solution of the system

$$\int f(t,x)\rho_t(dx) - \int f(0,x)\rho_0(dx) =$$

$$\int_0^t \int [B(x,s)\nabla f(x) + Lf(x) + \partial_s f(s,x)]\rho_s(dx)ds$$

for all $t \geq 0$, $f \in C_b^{2,1}(\mathbf{R}^d \times \mathbf{R}_+)$.

Then for any $t > 0$, $\rho_t(dx)$ has a density $\rho_t(x)$ with respect to Lebesgue measure, which satisfies for all $s, t \leq T$,

$$||\rho_t(x)||_\infty \leq c(t^{-d/2} + 1) \qquad (E_1)$$

$$|\rho_t(x) - \rho_t(y)| \leq c((t \wedge s)^{-(d+1)/2} + 1)|x - y|^{1/2} \qquad (E_2)$$

$$|\rho_t(x) - \rho_s(x)| \leq c((t \wedge s)^{-(d+1)/2} + 1)|t - s|^{1/5} \qquad (E_3)$$

where c is some constant which depends on $d, ||B||_\infty, T$.

Before giving the proof we recall certain results concerning upper bounds for $G(t, x, y)$ and its derivatives. [10], [11], [12] see also ([16], Proposition 2). We recall that

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(x) \partial_{x_i}$$

and the coefficients $a, b \in C_b^{2+\alpha}(\mathbf{R}^d)$.

Proposition 4.2.
1) For any $T > 0$ there exist some constants $c_1, c_2 > 0$ such that

$$t \sum_{i,j=1}^{d} |\partial^2_{x_i x_j} G(t, x, y)| + t^{1/2} \sum_{i=1}^{d} |\partial_{x_i} G(t, x, y)|$$

$$+ G(t, x, y) \leq c_1 t^{-d/2} \exp(-c_2 \frac{|x - y|^2}{2t})$$

2) The functions $\partial_{y_j} G(t, x, y), \partial^2_{x_i y_j} G(t, x, y)$ exist and are continuous functions of (t, x, y) in $(0, +\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. Moreover for all $T > 0$ there exist $c_1, c_2 > 0$ such that

$$t \sum_{i,j=1}^{d} |\partial^2_{x_i y_j} G(t, x, y)| + t^{1/2} \sum_{i=1}^{d} |\partial_{y_i} G(t, x, y)| \leq c_1 t^{-d/2} \exp(-c_2 \frac{|x - y|^2}{2t}).$$

Now we come back to the proof of Proposition 4.1.

Proof. We follow the proof in ([14], proposition 3.4.). Let $G(t, x, y)$ be the transition density of L. For any $\gamma \in L^1(\mathbf{R}^d), t \in (0, T]$ and $h > 0$ the function

$$\gamma_h(x, s) := \int \gamma(y) G(t + h - s, x, y) dy$$

is in $C_b^{2,1}(\mathbf{R}^d \times [0, t])$. It satisfies

$$\partial_s \gamma_h(x, s) + L \gamma_h(x, s) = 0$$

and therefore

$$\int \gamma_h(x,t))\rho_t(dx) - \int \gamma_h(x,0))\rho_0(dx) = \int_0^t \int B(x,s)\nabla\gamma_h(x,s)\rho_s(dx)ds \tag{18}$$

The equality (18) applied for $|\gamma|$ and the upper bounds on G imply

$$|\int \gamma_h(x,t)\rho_t(dx)| \leq \int\int |\gamma(y)|G(h,x,y)dy\rho_t(dx)$$

$$\leq \int\int |\gamma(y)|G(t+h,y,x)dy\rho_0(dx) + c\int_0^t \int\int |\gamma(y)||\nabla_x G(t+h-s,x,y)|dy\rho_s(dx)ds$$

$$\leq c\int\int |\gamma(y)|\frac{1}{(t+h)^{d/2}}\exp(-c_2\frac{|x-y|^2}{2(t+h)})dy\rho_0(dx) +$$

$$c\int_0^t \int\int |\gamma(y)||\nabla_x G(t+h-s,y,x)|dy\rho_s(dx)ds$$

$$\leq c[(t+h)^{-d/2}|\gamma|_1 + \int_0^t \int (t+h-s)^{-1/2}G(2(t+h-s),y,x)|\gamma(y)|dy\rho_s(dx)ds] \tag{19}$$

thanks to upper bound of $\nabla G(t,x,y)$ and (17).

In the other way from upper bounds of G we have the following estimate

$$\int_0^t \int (t+h-s)^{-1/2}G(2(t+h-s),y,x)|\gamma(y)|dy\rho_s(dx)ds \leq$$

$$c\int_0^t \int (t+h-s)^{-1/2}(t+h-s)^{-d/2}|\gamma(y)|dy\rho_s(dx)ds$$

$$\leq c|\gamma|_1 \int_0^t (t+h-s)^{-(d+1)/2}ds \leq c|\gamma|_1[h^{-(d-1)/2} + \delta_d^1|\ln(h)|]$$

where $\delta_i^j = 1$ if $i=j$ and 0 if not. In a first step we get for all $t \in (0,T], h \in (0,1)$,

$$|\int\int \gamma(y)G(h,y,x)dy\rho_t(dx)| \leq c|\gamma|_1[(t+h)^{-d/2} + h^{-(d-1)/2} + \delta_d^1|\ln(h)| + 1]. \tag{20}$$

Inserting the latter estimate in (19) we obtain

$$\int\int |\gamma(y)|G(h,y,x)dy\rho_t(dx) \leq$$

$$c|\gamma|_1[(t+h)^{-d/2} + \int_0^t (t+h-s)^{-1/2}((s+2(t+h-s))^{-d/2} +$$
$$(t+h-s)^{-(d-1)/2} + \delta_d^1|\ln(t+h-s)| + 1)ds] \leq$$
$$c|\gamma|_1[(t+h)^{-d/2} + h^{-(d-2)/2} + \delta_d^2|\ln(h)| + 1].$$

The exponent of $1/h$ is $(d-2)/2$ is less than the exponent of $1/h$ in the first estimate (20). We continue by inserting (20) in (19) as in [14] (Proposition 3.4.) until the d-th step. Finally we get

$$\int [\int |\gamma(y)|G(h,x,y)dy]\rho_t(dx) \leq c(t^{-d/2}+1)|\gamma|_1$$

uniformly in h, which implies that

$$|| \int G(h,x,\cdot)\rho_t(dx)||_\infty \leq c(t^{-d/2}+1)$$

uniformly in h. Since $\int G(h,x,y)\rho(dy,t) \to \rho(dx,t)$ weakly as $h \to 0+$, we obtain for all any open set $G \subset \mathbf{R}^d$ with finite Lebesgue measure $|G|$

$$(\rho_t(dx), 1_G) \leq \liminf_{h\to 0}(\int G(h,y,\cdot)\rho_t(dy), 1_G)$$

$$= \liminf_{h\to 0}(\rho_t(dx), \int_D G(h,y,\cdot)dy) \leq c(t^{-d/2}+1)|G|.$$

The proof of the fact that the density $\rho_t(x)$ of $\rho_t(dx)$ satisfies, for all $t \in (0,T]$,

$$||\rho_t(\cdot)||_\infty \leq c(t^{-d/2}+1)$$

is the same as in [14]. For the sake of completeness we recall it. For $\varepsilon > 0$ let

$$B_{\varepsilon,t} = \{x : \rho_t(x) > c(t^{-d/2}+1)(1+\varepsilon)\}.$$

If $|B_{\varepsilon,t}| > 0$, then there exists an open set A such that

$$B_{\varepsilon,t} \subset A,\ |A \setminus B_{\varepsilon,t}| \leq |A|\varepsilon/2.$$

The inequality

$$|A|c(t^{-d/2}+1) < |A|c(t^{-d/2}+1)(1+\varepsilon)(1-\varepsilon/2)$$

$$= c(t^{-d/2}+1)(1+\varepsilon)(|A| - |A|\varepsilon/2)c(t^{-d/2}+1)(1+\varepsilon)(|A| - |A|\varepsilon/2)$$
$$\leq c(t^{-d/2}+1)(1+\varepsilon)(|A| - |A \setminus B_{\varepsilon,t}|)$$

$$= c(t^{-d/2}+1)(1+\varepsilon)|B_{\varepsilon,t}|$$

$$\leq \int_{B_{\varepsilon,t}} \rho_t(x)dx \leq \int_A \rho_t(x)dx \leq |A|c(t^{-d/2}+1)$$

proves that

$$\rho_t(x) \leq c(t^{-d/2}+1),$$

which achieves the proof of (E_1).

Now we prove the estimate (E_2). Let $p = t/2, y, y' \in \mathbf{R}^d$ with $\delta = |y-y'| < 1 \wedge t$. We have from (18)

$$|\int \rho_t(x)G(h,x,y)dx - \int \rho_t(x)G(h,x,y')dx|^2 =$$

$$|\int \rho_p(x)[G(t-p+h,x,y) - G(t-p+h,x,y')]dx +$$

$$\int_p^t \int \rho_s(x)B(x,s)[\nabla_x G(t+h-s,x,y) - \nabla_x G(t+h-s,x,y')]dxds|^2$$

$$\leq 3|\int \rho_p(x)[G(t-p+h,x,y) - G(t-p+h,x,y')]dx|^2 +$$

$$3|\int_p^{t-\delta} \int \rho_s(x)B(x,s)[\nabla_x G(t+h-s,x,y) - \nabla_x G(t+h-s,x,y')]dxds|^2 +$$

$$3|\int_{t-\delta}^t \int \rho_s(x)B(x,s)[\nabla_x G(t+h-s,x,y) - \nabla_x G(t+h-s,x,y')]dxds|^2.$$

From the upper bound of $\nabla_y G(t,x,y)$ we derive

$$|\int \rho_p(x)[G(t-p+h,x,y) - G(t-p+h,x,y')]dx|$$

$$\leq |y-y'| \int \rho_p(x)|\nabla_z G(t-p+h,x,z)|dx \leq c|y-y'|t^{-\frac{d+1}{2}}$$

From the upper bound of $\partial^2_{x_i y_j} G(t,x,y)$ and the estimate $||\rho_s||_\infty \leq c(1+t^{-d/2}) \leq c(1+t^{-(d+1)/2})$ for all $s \in (p,t)$ and for all $t \in (0,T]$, we have

$$\int_p^{t-\delta} \int \rho_s(x)B(x,s)[\nabla_x G(t+h-s,x,y) - \nabla_x G(t+h-s,x,y')]dxds \leq$$

$$c|y-y'|(1+t^{-(d+1)/2}) \int_p^{t-\delta} \int (t-s)^{-1}ds$$

$$\leq c|y-y'|(1+t^{-(d+1)/2})[|\ln(\delta)|+1]$$

It remains the term

$$|\int_{t-\delta}^{t}\int \rho_s(x)B(x,s)[\nabla_x G(t+h-s,x,y)-\nabla_x G(t+h-s,x,y')]dxds| \leq$$

$$c\int_{t-\delta}^{t}(t+h-s)^{-1/2}(1+t^{-d/2})ds \leq c(1+t^{-d/2})\delta^{1/2}$$

thanks to the upper bound of $\partial_{x_i}G(t,x,y)$. Finally we get

$$|\int \rho_t(x)G(h,x,y)dx-\int \rho_t(x)G(h,x,y')dx|^2$$

$$\leq c[|y-y'|^2(1+|\ln(\delta)|^2)+\delta](t^{-(d+1)/2}+1)^2$$
$$\leq c|y-y'|(t^{-(d+1)/2}+1)^2$$

because $|y-y'|\leq 1$.

This estimate holds for any fixed $t\in(0,T]$ uniformly in $h>0$, and therefore is valid for the weak limit of the function $y\to\int\rho_t(x)G(h,y,x)dx$, as $h\to 0$, namely

$$|\rho_t(y)-\rho_t(y')|\leq c|y-y'|^{1/2}(t^{-(d+1)/2}+1)$$

for $|y-y'|\leq 1$. As $|\rho_t(y)-\rho_t(y')|\leq c(1+t^{-(d+1)/2})$ for all y,y', then we can derive that

$$|\rho_t(y)-\rho_t(y')|\leq c|y-y'|^{1/2}(t^{-(d+1)/2}+1)$$

for all y,y'.

We still have to establish (E_3). For $0<s<t, h=|t-s|^{4/5}$ we obtain

$$|\rho_t(y)-\rho_s(y)|\leq|\rho_t(y)-\int\rho_t(x)G(h,x,y)dx|+$$

$$|\int[\rho_t(x)-\rho_s(x)]G(h,x,y)dx|+$$

$$|\int[\rho_s(x)-\rho_s(y)]G(h,x,y)dx|\leq I_1+I_2+I_3$$

The term

$$I_1=|\int[\rho_t(y)-\rho_t(x)]G(h,x,y)dx|\leq c\int|y-x|^{1/2}G(h,x,y)dx(t^{-(d+1)/2}+1)$$

$$c\int|y-x|^{1/2}\frac{1}{h^{d/2}}\exp(-c_2\frac{|x-y|^2}{2h})dx(t^{-(d+1)/2}+1)$$

$$\leq c(t^{-(d+1)/2}+1)h^{1/4}.$$

Similarly

$$I_3 \leq c(s^{-(d+1)/2}+1)h^{1/4}.$$

Furthermore

$$I_2 = |\int [\rho_t(x)-\rho_s(x)]G(h,x,y)dx| = |\int_s^t \int \rho_r(x)[B(x,r)\nabla_x G(h,x,y)+$$

$$LG(h,x,y)]dxdr|$$

$$\leq c|t-s|(s^{-d/2}+1)[(h)^{-1/2}+h^{-1}] \leq c|t-s|^{1/5}(s^{-d/2}+1)$$

4.2 Proof of uniqueness

We are going to give the uniqueness in any interval $[0,T]$. The proof is similar to the uniqueness in [4]. Let $X_t = X_0 + \int_0^t (\mathbf{E}[v(X_0)\,|\,X_s\,]+b(X_s))ds + \int_0^t \sigma(X_s)B_t$ be a weak solution. We have already shown in Section 3 that

$$(\rho(x,t)dx = P(X_t \in dx), u(x,t) = \mathbf{E}[v(X_0)\,|\,X_t = x\,])$$

is a weak solution of $\mathcal{S}(a,b)$. It is easy to show that

$$q_i(x,t) := u_i(x,t)\rho(x,t) = \int v_i(y)\rho(0,y,t,x)\rho_0(dy), \tag{21}$$

where $\rho(0,y,t,x)$ is the fundamental solution of

$$\partial_t(\rho) + div(\rho u) = L^*(\rho).$$

Let $q_0(x,t) = q_0(x,t,+) = q_0(x,t,-) := \rho(x,t)$,

$$q_i(x,t,+) = \int v_i^+(y)\rho(0,y,t,x)\rho_0(dy),$$

and

$$q_i(x,t,-) = \int v_i^-(y)\rho(0,y,t,x)\rho_0(dy)\,\forall\, 1 \leq i \leq d,$$

where x^+, x^- denote respectively the positive and the negative parts of the real number x.

Let $\varepsilon = +,-$, and $\frac{1}{c_i^\varepsilon} = \int v_i^\varepsilon(y)\rho_0(dy)$. We can show that for each couple (i,ε), $(t \to c_i^\varepsilon q_i(x,t,\varepsilon) := m(x,t))$ is a family of probability density on $\mathbf{R}^d$ which is a weak solution of the parabolic equation

$$\partial_t(m) + div(mB) = L^*(m),$$

where

$$B(x,t) = \frac{1}{q_0(x,t)}(q_1(x,t), ..., q_d(x,t)).$$

It follows from Subsection 4.1 that there exists a constant c which depends only on $||v||_\infty$ and d such that for each $i = 0, ..., d, \varepsilon = +, -$

$$||q_i(\cdot, t, \varepsilon)||_\infty \leq c(t^{-d/2} + 1),$$

$$|q_i(y, t, \varepsilon) - q_i(y', t, \varepsilon)| \leq c(t^{-(d+1)/2} + 1)|y - y'|^{1/2},$$

$$|q_i(y, t, \varepsilon) - q_i(y, s, \varepsilon)| \leq c(\min(t, s)^{-(d+1)/2} + 1)|t - s|^{1/5}.$$

It also follows that

$$q = (q_0(x,t), q_1(x,t,+), q_1(x,t-), ..., q_d(x,t,+), q_d(x,t,-)) :=$$

$$(q_i(\varepsilon) : 0 \leq i \leq d, \varepsilon = +, -)$$

is a weak solution of the following $2d+1$-dimensional parabolic system

$$\mathcal{P}(2d+1) \begin{cases} \partial_t(q_i(\varepsilon)) + div(q_i(\varepsilon)F(q)) = L^*(q_i(\varepsilon)), \\ \forall\, 0 \leq i \leq d,\ \varepsilon = +, -, \\ q_i(\varepsilon, dx, t) \to q_i(\varepsilon, dx, 0) \\ \text{weakly, as } \ t \to 0^+. \end{cases}$$

where

$$F(q) = \frac{1}{q_0}(q_1(+) - q_1(-), ..., q_d(+) - q_d(-)).$$

We point out that each $q \to q_i(\varepsilon)F(q)$ is Lipshitz continuous on the domain

$$D = \{q \in \mathbf{R}_+ \times \mathbf{R}^{2d} : |q_i(+)| \leq ||v||_\infty q_0, |q_i(-)| \leq ||v||_\infty q_0, \forall\, 1 \leq i \leq d\}.$$

In the sequel we denote $|q| := q_0 + \sum_{i=0}^d |q_i(+)| + |q_i(-)|$ the norm of the vector q in $\mathbf{R}_+ \times \mathbf{R}^{2d}$.

Now we are ready to prove the uniqueness by mimicking the method of *Oelschläger* ([14], Proposition 3.5). Let us consider two weak solutions $(q_i^1, 0 \leq i \leq d), (q_i^2, 0 \leq i \leq d)$ of $S(a,b)$ with the same initial conditions $(\rho_0(dx), q_0(dx))$. We are going to show that

$$q_0^1(x,t) = q_0^2(x,t), q_i^1(x,t,+) = q_i^2(x,t,+), q_i^1(x,t,-) = q_i^2(x,t,-)\ \forall\, 1 \leq i \leq d.$$

From the system $\mathcal{P}(2d+1)$ we have for all $f \in C_b^{2,1}(\mathbf{R}^d \times [0,T])$, and for $i = 0, ..., d,\ j = 1, 2,$

$$\int f(x,t) q_i^j(x,t,+) dx - \int f(x,0) q_i^j(dx, 0) =$$

$$\int_0^t \int [F(q^j)\nabla f(x,s) + Lf(x,s) + \partial_s f(x,s)] q_i^j(x,s,+) dxds \tag{22}$$

For any $h > 0$ the function $\gamma_{t+h-s}(x) := \int G(t+h-s,x,y)\gamma(y)dy$ is in $C_b^{2,1}(\mathbf{R}^d \times [0,t])$ for all $\gamma \in L^1(\mathbf{R}^d)$. Therefore (22), and

$$\partial_s G(s,y,x) - LG(s,y,x) = 0, \text{on } \mathbf{R}^d \times (0,+\infty), \tag{23}$$

yields

$$|q_i^1(y,t,+) - q_i^2(y,t,+)| =$$
$$|\int_0^t \int [F(q^1(z,s))\nabla_z G(t-s,z,y) q_i^1(z,s,+) -$$
$$F(q^2(z,s))\nabla_z G(t-s,z,y) q_i^2(z,s,+)] dzds|.$$

Now we multiply both sides of this equation with the function $G(h,y,x), h > 0$ and we integrate them. From this and from (23) we get the inequality

$$\int |q_i^1(y,t,+) - q_i^2(y,t,+)| G(h,y,x) dy \leq$$

$$\int_0^t \int \int |F(q^1(z,s)) q_i^1(z,s,+) - F(q^2(z,s)) q_i^2(z,s,+)|$$
$$|\nabla_z G(t-s,z,y)| G(h,y,x) dzdyds$$
$$\leq c \int_0^t \int \int |q^1(z,s) - q^2(z,s)| (t-s)^{-1/2}$$
$$G(2(t-s),z,y) G(h,y,x) dzdyds.$$

Noting that

$$\int G(t,z,y) G(s,y,x) dy = G(t+s,z,x)$$

It follows that

$$\int |q_i^1(y,t,+) - q_i^2(y,t,+)| G(h,y,x) dy \leq$$

$$c \int_0^t \int |q^1(z,s) - q^2(z,s)| (t-s)^{-1/2} G(2(t-s)+h,z,x) dzds,$$

where c is some constant which depends on $d, ||v||_\infty, T$. The same estimates show that for all $0 \leq i \leq d$

$$\int |q_i^1(y,t,-) - q_i^2(y,t,-)| G(h,y,x) dy \leq$$

$$c\int_0^t\int |q^1(z,s)-q^2(z,s)|(t-s)^{-1/2}G(2(t-s)+h,z,x)dzds.$$

We derive that

$$\int |q^1(y,t)-q^2(y,t)|G(h,y,x)dy \leq$$

$$c\int_0^t\int |q^1(z,s)-q^2(z,s)|(t-s)^{-1/2}G(2(t-s)+h,z,x)dzds,$$

this means with

$$Q(h,t,x)=\int |q^1(y,t)-q^2(y,t)|G(h,y,x)dy,$$

that

$$Q(h,t,x)\leq c\int_0^t (t-s)^{-1/2}Q(2(t-s)+h,s,x)ds. \tag{24}$$

Now the proof goes as in ([14] page 305). For the sake of completeness we recall it. From the estimate $G(h,y,x)\leq \frac{c}{h^{d/2}}$ we get a first estimate

$$Q(h,t,x)\leq ch^{-d/2}. \tag{25}$$

Inserted into (24) gives uniformly in $t\in(0,T], x\in\mathbf{R}^d$

$$\begin{aligned} Q(h,t,x) &\leq c\int_0^t (t-s)^{-1/2}(2(t-s)+h)^{-d/2}ds \\ &\leq c(h^{-(d-1)/2}+\delta_d^1|\ln(h)|+1). \end{aligned} \tag{26}$$

The latter estimate is an improvement of (25) and if we insert this improvement into (24) we obtain uniformly in $t\in(0,T], x\in\mathbf{R}^d$

$$Q(h,t,x)\leq c(h^{-(d-2)/2}+\delta_d^2|\ln(h)|+1).$$

Continuing in this way we finally obtain uniformly in $t\in(0,T], x\in\mathbf{R}^d$

$$Q(h,t,x)\leq c$$

Since $G(h,y,x)\to\delta_y(x)$ as $h\to 0+$, then

$$Q(h,t,x)\to |q^1(x,t)-q^2(x,t)|$$

this yields

$$\sup_{x\in\mathbf{R}^d,t\in(0,T]}|q^1(x,t)-q^2(x,t)|<c. \tag{27}$$

Now recall that for each i,

$$|q_i^1(y,t,+) - q_i^2(y,t,+)| =$$

$$|\int_0^t \int [F(q^1(x,s))\nabla_x G(t-s,x,y) q_i^1(x,s,+) -$$

$$F(q^2(x,s))\nabla_x G(t-s,x,y) q_i^2(x,s,+)] dx ds|$$

$$\leq c \int_0^t \sup_{z \in \mathbf{R}^d, s \leq T} |q^1(z,s) - q^2(z,s)| (t-s)^{-1/2} ds$$

$$\leq c \sup_{z \in \mathbf{R}^d, s \leq \tau} |q^1(z,s) - q^2(z,s)| \sqrt{\tau}$$

uniformly in $t \in (0,\tau], y, \tau \leq T$, and therefore

$$\sup_{z \in \mathbf{R}^d, s \leq \tau} |q^1(z,s) - q^2(z,s)| \leq c \sup_{z, s \leq \tau} |q^1(z,s) - q^2(z,s)| \sqrt{\tau}$$

For $\tau < 1/c^2$ this yields by (27)

$$\sup_{z \in \mathbf{R}^d, s \leq \tau} |q^1(z,s) - q^2(z,s)| = 0$$

Iterating of the above argument in $[\tau, 2\tau]$, etc. provides the desired result, namely

$$\sup_{z \in \mathbf{R}^d, s \leq T} |q^1(z,s) - q^2(z,s)| = 0$$

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